数理解析研究所講究録
342巻 1979年 1-4



# Note on shape theory II : Problems

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Let Ⓜ be the class consisting of metrizable spaces. We define a shape category in M called a fine shape category as follows. Let M and N be spaces in Ⓜ and let X and Y be closed sets in M and N respectively. According to [7] a continuous map f : M-X $\longrightarrow$ N is said to be a <u>fine map</u> from M-X into N rel.X,Y if for every neighborhood V of Y in N there is a neighborhood U of X in M such that f(U-X) $\subset$ V. Two fine maps f,g : M-X $\longrightarrow$ N rel. X,Y are <u>fine homotopic</u> (notation: $f \underset{F}{\simeq} g$ rel.X,Y) if there is a homotopy H : (M-X) $\times$ I $\longrightarrow$ N satisfying the following condition: H(x,0) = f(x), H(x,1) = g(x) for x $\in$ M-X, and for every neighborhood V of Y in N there is a neighborhood U of X in M such that H((U-X) $\times$ I) $\subset$ V. The following lemmas are obvious.

Lemma 1. If Y is an unstable closed set of N and f : M-X $\longrightarrow$ N is a fine map rel.X,Y, then there is a fine map f' : M-X $\longrightarrow$ N rel.X,Y such that f'(M-X) $\subset$ N-Y and $f \underset{F}{\simeq} f'$ rel.X,Y.

Lemma 2. Let L $\in$ Ⓜ and Z a closed set of L. If f : M-X $\longrightarrow$ N is a fine map rel.X,Y such that f(M-X) $\subset$ N-Y and g : N-Y $\longrightarrow$ L is a fine map rel.Y,Z, then gf : M-X $\longrightarrow$ L is a fine map rel.X,Z.

A fine map f : M-X $\longrightarrow$ N rel.X,Y is said to be a <u>fine equivalence</u> if there are fine maps f' : M-X $\longrightarrow$ N rel.X,Y and g : N-Y $\longrightarrow$ M rel.Y,X such that f'(M-X) $\subset$ N-Y, $f' \underset{F}{\simeq} f$ rel.X,Y, and

(1) $$gf' \underset{F}{\simeq} 1_{M-X} \quad \text{rel.X,X} \quad ,$$

(2) $\quad f'g \underset{F}{\simeq} 1_{N-Y} \quad \text{rel.}Y,Y \quad ,$

where $1_{M-X}$ and $1_{N-Y}$ are the identity fine maps in M-X and N-y respectively. If only (1) is satisfied, then f is said to be a _fine domination_.

For $X,Y \in$ Ⓜ, let M and N be AR's containing X,Y as unstable closed sets respectively. If there is a fine equivalence f : $M-X \longrightarrow N$ rel.X,Y, then we say that X and Y are _fine shape equivalent_ or have the same _fine shape_ and write $Sh_F(X) = Sh_F(Y)$. If there is a fine domination f : $M-X \longrightarrow N$, then X _fine shape dominates_ Y and we write $Sh_F(X) \geqq Sh_F(Y)$. By Lemmas 1 and 2, it is easy to see that there is a shape category consisting of spaces in M whose morphisms are the fine homotopy equivalence classes of fine maps. We call it a _fine shape category_. If spaces are compact then the fine shape defined here is the same as one defined in [9].

Problem 1. For $X \in$ Ⓜ, what relation is there between Sh(X) and $Sh_F(X)$ or $Sh_W(X)$ and $Sh_F(X)$ ? If X and Y are locally compact and $Sh(X) \geqq Sh(Y)$ or $Sh_W(X) \geqq Sh_W(Y)$, is $Sh_F(X) \geqq Sh_F(Y)$ true ? Here Sh(X) is the shape of X in the sense of Fox [4] and $Sh_W(X)$ is the weak shape in the sense of Borsuk [1].

Problem 2. Characterize the space X such that $Sh_F(X) = 1$, that is, X is fine shape equivalent with a one point space.

The following are problems concerning the shape Sh(X) however many of them are interesting one's concerning the fine shape $Sh_F(X)$ too.

Problem 3. Let $X, Y, X', Y' \in$ Ⓜ and let $Y, Y'$ be locally compact. If $Sh(X) = Sh(X')$ and $Sh(Y) = Sh(Y')$, is $Sh(X \times Y) = Sh(X' \times Y')$ true ?

Problem 4. Are there $X \in$ Ⓜ and a locally compact space $Y \in$ Ⓜ such that $Fd(X \times Y) < Fd(X) + Fd(Y)$ ? Here $Fd(X) = \mathrm{Min}\{\dim Y : Y \in$ Ⓜ and $Sh(X) \leqq Sh(Y)\}$.

Cf. [5] and [6].

Problem 5. (S.Mardešić) Let $X \in$ Ⓜ and let $Y$ be a locally finite simplicial polytope.

(1) Is there the product $Sh(X) \times Sh(Y)$ ?

(2) If $Sh(X) \times Sh(Y)$ exists, is $Sh(X) \times Sh(Y) = Sh(X \times Y)$ ?

Problem 6. Let $X, Y \in$ Ⓜ and $f : X \to Y$. If there is a locally finite open cover Ⓤ $= \{U_\alpha : \alpha \in \Lambda\}$ such that for any finite set $\{\alpha_0, \cdots, \alpha_n\} \subset \Lambda$ $f \mid f^{-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}) : f^{-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}) \longrightarrow U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$ is a shape equivalence, is $f$ a shape equivalence ?

By the same way as in the proof of [8], the problem is solved affirmatively for the fine shape $Sh_F$.

Problem 7. If $X$ is a compactum which is a 1-1 continuous image of a locally compact ANR, then:

(1) Is $X$ an FANR ?

(2) Is $X$ movable ?

Problem 8. Let $X$ be a locally compact space in M. If $X$ is movable in the sense of Kozlowski and Segal [10], is every metrizable compactification of $X$ movable ?

Problem 9. Let $X$ be a compactum. If there is a countable number of movable compacta $X_i$, $i=1,2,\ldots$, such that $X = \bigcup_{i=1}^{\infty} X_i$, is X movable ? (Here $X_i$ and $X_j$, $i\neq j$, are not necessarily disjoint.)

Problem 10. Let $(X,x_0)$ and $(Y,y_0)$ be pointed compacta. If $\underline{f} : (X,x_0) \longrightarrow (Y,y_0)$ is a shaping such that $\underline{f}_* : \underline{\underline{\pi}}_n(X,x_0) \xrightarrow{\approx} \underline{\underline{\pi}}_n(Y,y_0)$ for $n = 0,1,2,\ldots$, is $Sh_F(X,x_0) = Sh_F(Y,y_0)$ true ? Here $\underline{\underline{\pi}}_*$ is the shape group defined by Quigley [11].

Problem 11. (Chapman) Is every weak proper homotopy equivalence a proper homotopy equivalence ?

If Problem 11 has an affirmative solution, the second part of Problem 1 is so. Cf. [2] and [3].

## References


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